Recent Math Problems Arising in Statistics involving Generalized Functions, Group Representations, Inequalities, and the Riemann Zeta Function

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Paul Dirac (1902-1984)
Sergi Sobolov (1908-1989) and Laurent Schwartz (1915-2002)
Let
\[ f : \mathbb{R} \to \mathbb{R} \]
be a locally integrable function and let
\[ \varphi : \mathbb{R} \to \mathbb{R} \]
be infinitely differentiable with compact support.
Set
\[ \langle f, \varphi \rangle = \int_{\mathbb{R}} f(x) \varphi(x) \, dx \]
\(f\) can therefore be viewed as a continuous linear functional on test functions \(\varphi\). The derivative of the distribution \(S\) is naturally defined to be
\[ \langle S', \varphi \rangle \triangleq -\langle S, \varphi' \rangle \]
The **Fourier transform** can be defined on *tempered* distributions. Test functions are from the **Schwartz space** — infinitely differentiable rapidly decreasing functions — $\forall n, m$

\[
\sup_{x \in \mathbb{R}} |x^n \phi^{(m)}(x)| < \infty
\]

\[
\tilde{\phi}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(s) e^{-isx} \, ds
\]

\[
\langle \tilde{S}, \phi \rangle = \langle S, \tilde{\phi} \rangle
\]
Let \( X_1, X_2, \ldots, X_n \overset{iid}{\sim} F(x) \). The survival function, \( S(x) = 1 - F(x) \), has estimate

\[
\hat{S}(t) = 1 - \hat{F}(t) = 1 - \frac{1}{n} \sum_{j=1}^{n} I(X_i \leq t)
\]
**Bias of Smoothed EDF**

\[
\hat{F}_h(t) = \int_{-\infty}^{t} \hat{f}_h(x) \, dx = \frac{1}{n} \sum_{j=1}^{n} \bar{K}\left(\frac{t-X_j}{h}\right)
\]

where \(\bar{K}(t) = \int_{-\infty}^{t} K(x) \, dx\).

\[
E[\hat{F}_h(t)] = \frac{1}{n} \sum_{j=1}^{n} E\left[\bar{K}\left(\frac{t-X_i}{h}\right)\right],
\]

\[
E\left[\bar{K}\left(\frac{t-X_i}{h}\right)\right] = \int_{-\infty}^{\infty} \bar{K}\left(\frac{t-x}{h}\right) f(x) \, dx
\]

\[
= \int_{-\infty}^{\infty} \bar{K}\left(\frac{t-x}{h}\right) dF(x)
\]

\[
= \bar{K}\left(\frac{t-x}{h}\right) F(x) \bigg|_{x=\infty}^{x=\infty} + \frac{1}{h} \int F(x) K\left(\frac{t-x}{h}\right) \, dx
\]

\[
= 0
\]

\[
= F \ast K_h(t)
\]

where \(K_h(t) = \frac{1}{h} K\left(\frac{t}{h}\right)\).
Fourier Transform of a CDF

Note that

\[ F(t) = \int_{-\infty}^{t} f(x) \, dx = \int_{-\infty}^{\infty} f(x) H(t - x) \, dx = f \ast H(t) \]

where \( H(x) \) is the Heaviside step function given by \( H(x) = 1(x > 0) \). Therefore

\[ \mathcal{F} (F(t)) = \phi(s) \left( \pi \delta(s) + \frac{1}{is} \right) \]

\[ = \pi \phi(0) \delta(s) + \frac{\phi(s)}{is} \]

\[ = \pi \delta(s) + \frac{\phi(s)}{is}. \]

where \( \phi(s) \) is the characteristic function (the Fourier transform of \( f \)).
Bias Calculation Continued

\[
\text{bias } (\hat{F}_h(t)) = K_h \ast F(t) - F(t)
\]
\[
= \mathcal{F} \left( \mathcal{F}^{-1} (K_h \ast F(t) - F(t)) \right)
\]
\[
= \mathcal{F} \left( \mathcal{F}^{-1} (K_h) \cdot \mathcal{F}^{-1} (F) - \mathcal{F}^{-1} (F) \right)
\]
\[
= \mathcal{F} \left( (\mathcal{F}^{-1} (K_h) - 1) \mathcal{F}^{-1} (F) \right)
\]
\[
= \mathcal{F} \left( (\kappa (sh) - 1) \left( \pi \delta(s) + \frac{\phi(s)}{is} \right) \right)
\]
\[
= \mathcal{F} \left( (\kappa (sh) - 1) \frac{\phi(s)}{is} \right) - \pi \mathcal{F} \left( (\kappa (sh) - 1) \delta(s) \right)
\]
\[
= \mathcal{F} \left( (\kappa (sh) - 1) \frac{\phi(s)}{is} \right) - \pi \mathcal{F} \left( (\kappa (sh) - 1) \bigg|_{s=0} \right)
\]
\[
= 0
\]
\[
= \frac{1}{2\pi} \int_{|s| > 1/h} (\kappa (sh) - 1) \frac{\phi(s)}{is} \, ds.
\]
Symmetries of the Bivariate ACF and Bispectrum
Symmetries of ACF

One variable: \( C(x) = C(-x) \)

Two variables:

\[
C(x, y) \overset{1}{=} C(x, y) \\
\overset{2}{=} C(-x, y - x) \\
\overset{3}{=} C(y, x) \\
\overset{4}{=} C(x - y, -y) \\
\overset{5}{=} C(-y, x - y) \\
\overset{6}{=} C(y - x, -x)
\]

Eq. 2 + Eq. 3 ⇒ Eq. 6

\[ C(x, y) = C(y, x) = C(-x, y - x) \text{ suffices} \]

⋆ We wish 1) understand the symmetries and 2) symmetrize a non-symmetric function.
Constructing the Symmetries from Permutations

★Each equation corresponds to a permutation.

For simplicity, assume $E[X_t] = 0$.

The equation corresponding to the permutation $\sigma = (12)$ is

$$C(x, y) = E[X_tX_{t+x}X_{t+y}]$$

$$\overset{(12)}{=} E[X_{t+x}X_tX_{t+y}]$$

$$= E[X_tX_{t-x}X_{t-x+y}]$$

$$= C(-x, y - x)$$
A Group Representation

Define $\psi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ given by $\psi(a, b, c) \mapsto (b - a, c - a)$. Take $\sigma = (12)$, then

$$(x, y) \xrightarrow{\sigma} (x, 0, y) \xrightarrow{\psi} (-x, y - x) \mapsto \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$e \longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \longleftrightarrow C(x, y) \quad (13) \longleftrightarrow \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \longleftrightarrow C(x - y, -y)$$

$$(12) \longleftrightarrow \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} \longleftrightarrow C(-x, y - x) \quad (123) \longleftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \longleftrightarrow C(-y, x - y)$$

$$(23) \longleftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \longleftrightarrow C(y, x) \quad (132) \longleftrightarrow \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \longleftrightarrow C(y - x, -x)$$

Suppose $\sigma = (12)$ and $\tau = (13)$, then $\gamma = (132) = \sigma \tau$ and

$$\rho(\gamma) = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \rho(\sigma) \rho(\tau)$$
Theorem (Berg 2007)

The mapping $\rho : S_n \rightarrow \text{GL}_{n-1}(\mathbb{R})$, described above, is a faithful group representation.

Symmetrizing lag-windows – generalizing current constructions

Current construction: $\tilde{f}(x, y) = f(x)f(y)f(x - y)$

$\tilde{f}(x, y) = h(f(x, y), f(-x, y-x), f(y, x), f(x-y, -y), f(-y, x-y), f(y-x, -x))$

where $h$ is any symmetric function of its six arguments.

Figure: $\tilde{f}$ with $h = \prod x_i$, $h = \max(x_i)$, $h = \min(x_i)$, and $f(x, y) = (1 - x^2 - y^2)^+$. 
Implications of the Representation II

2 Generalization of the Gabr-Rao optimal window

\[ \Lambda_{opt}(\omega) = \alpha \left( 1 - \beta \left( \sum_{i=1}^{k-1} \omega_i^2 + \sum_{i<j} \omega_i \omega_j \right) \right) ^+ \]

Theorem (Berg 2007)

Let \( \Lambda(\omega) \) be any nonnegative kernel that integrates to one and satisfies all the necessary symmetries. Also assume

\[ \int_{\mathbb{R}^{k-1}} \omega_j^2 \Lambda(\omega) \, d\omega = \int_{\mathbb{R}^{k-1}} \omega_j^2 \Lambda_{opt}(\omega) \, d\omega \]

for \( j = 1, \ldots, n - 1 \). Then \( \| \Lambda \|_{L_2} \geq \| \Lambda_{opt} \|_{L_2} \).
Agresti Example

<table>
<thead>
<tr>
<th></th>
<th>Lung Cancer</th>
<th>Heat Disease</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nonsmokers</td>
<td>$p_1 = .0001$</td>
<td>$p_3 = .00413$</td>
</tr>
<tr>
<td>Smokers</td>
<td>$p_2 = .0014$</td>
<td>$p_4 = .00669$</td>
</tr>
</tbody>
</table>

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<tr>
<th></th>
<th>Lung Cancer</th>
<th>Heat Disease</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relative Risk</td>
<td>14</td>
<td>1.62</td>
</tr>
<tr>
<td>Odds Ratio</td>
<td>14.02</td>
<td>1.62</td>
</tr>
<tr>
<td>Risk Difference</td>
<td>.00130</td>
<td>.00256</td>
</tr>
</tbody>
</table>

Formulas

$$RR = \frac{p_2}{p_1} \quad OR = \frac{p_2/(1 - p_2)}{p_1/(1 - p_1)} \quad RD = p_2 - p_1$$

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A General Result

Suppose $p_1, p_2, p_3, p_4 \in (0, 1)$ satisfy $p_1 + p_4 \leq 1$ or $p_2 + p_3 \leq 1$. Show that

$$\frac{p_2}{p_1} < \frac{p_4}{p_3} \quad (i)$$

and

$$p_2 - p_1 < p_4 - p_3 \quad (ii)$$

imply

$$\frac{p_2/(1 - p_2)}{p_1/(1 - p_1)} < \frac{p_4/(1 - p_4)}{p_3/(1 - p_3)} \quad (iii)$$

And also $(i)$ and $(ii)$ imply $(iii)$ when $<$ is replaced with $>$ in each of the inequalities.
Let $X_1, \ldots, X_n$ from a power law distribution

$$f(x) = f_\beta(x) = \frac{c}{x^\beta}; \quad x = 1, 2, \ldots$$

for some $\beta > 1$; note that $c = c_\beta = \frac{1}{\zeta(\beta)}$ where $\zeta$ is the Riemann zeta function.

Consider the problem of estimating $\beta$.

Define

$$H_n(x) = n\hat{f}(x) = \sum_{i=1}^{n} 1[X_i = x].$$

Natural estimate of $\beta$ given from a linear regression on

$$\{(\log x, \log H_n(x)) : x = 1, 2, \ldots\},$$
Optimal Least Squares

An optimal estimate of $\beta$ is a particular linear combination of $\hat{c}$ and $\hat{\beta}$. Supposing the regression routine produced the covariance matrix

$$
\begin{pmatrix}
\hat{\tau}_{11} & \hat{\tau}_{12} \\
\hat{\tau}_{21} & \hat{\tau}_{22}
\end{pmatrix}
$$

for the covariance of

$$
\begin{pmatrix}
\hat{\beta} \\
\hat{c}
\end{pmatrix}
$$

Note that

$$
\beta = \beta(c) = \zeta^{-1}(1/c) \quad \text{and} \quad \beta'(c) = \frac{-1}{c^2 \zeta'(\zeta^{-1}(1/c))}
$$

Basing an estimate of $\beta$ on $\tilde{\beta} = \zeta^{-1}(1/\hat{c})$, we have a correlated pair $\left(\hat{\beta}, \tilde{\beta}\right)$ of asymptotically unbiased estimates of $\beta$ with covariance efficiently estimated by

$$
\begin{pmatrix}
\hat{\tau}_{11} & \tilde{\tau}_{12} \\
\tilde{\tau}_{21} & \tilde{\tau}_{22}
\end{pmatrix} := \begin{pmatrix} 1 & 0 \\ 0 & \beta'(\hat{c}) \end{pmatrix} \begin{pmatrix}
\hat{\tau}_{11} & \hat{\tau}_{12} \\
\hat{\tau}_{21} & \hat{\tau}_{22}
\end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \beta'(\hat{c}) \end{pmatrix}
$$
There is an optimal convex combination $\hat{\beta}^* = a\hat{\beta} + (1 - a)\hat{\beta}$, with $a$ given by

$$a = \frac{\tilde{\tau}_{22} - \tilde{\tau}_{12}}{\tilde{\tau}_{11} - 2\tilde{\tau}_{12} + \tilde{\tau}_{22}},$$
Main Theorem

Theorem (Abramson, Berg, Meyers 2008)

Let $\beta > 1$ be fixed and $X_1, X_2, \ldots \overset{iid}{\sim} f(x) = cx^{-\beta} (x = 1, 2, \ldots)$. Define $H_n(x) = \sum_{i=1}^n 1[X_i = x]$ and $M_{nk} = M_n = \min\{x : H_n(x) \leq k\}$ where $k$ is any nonnegative integer. Provided a sequence $y_n$ satisfies

$$y_ne^{-ncy_n^{-\beta}} \rightarrow 0 \quad (\star)$$

as $n \to \infty$, it follows that $\Pr [M_n > y_n] \rightarrow 1$ as $n \to \infty$.

In estimating rates for $y_n$, we would like to approximate the difference between

$$\zeta(s) \quad \text{and} \quad \int_1^\infty x^{-s} \, dx$$

And show

$$\zeta(2) - 1 = \frac{\pi^2}{6} - 1 = \sup_{s \in (1,2]} \left[ \zeta(s) - \int_1^\infty x^{-s} \, dx \right] \quad (\star\star)$$